RELIABILITY APPROXIMATIONS FOR COMPLEX COHERENT SYSTEMS WITH HIGHLY RELIABLE COMPONENTS

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THESIS

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by

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ABSTRACT

Several reliability approximations for complex coherent systems with highly reliable components are reviewed and the need for some relatively simple approximations which would be useful in a preliminary reliability analysis of the same type of system is presented. Some approximations based on minimal paths, minimal cuts, most reliable minimal paths, and least reliable minimal cuts are investigated in terms of simplicity and accuracy. The possible applications of the approximations are illustrated through the use of examples.

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I. INTRODUCTION

During the past two decades, the world has experienced the most rapid technological advances ever known. Coupled with these advances have been the formation of extremely complex systems and the development of highly reliable system components. Computational difficulties in determining system reliability (the probability that a system will work under specified conditions) are proportional to system complexity and, consequently, have grown with system complexity. In fact, the computation of reasonably accurate approximations or bounds to system reliability has almost become a formidable problem in itself.

There exist a number of techniques for determining upper and lower bounds on system reliability. However, in many cases the computation of these bounds is nearly as difficult as determining the true system reliability. Relatively simple approximations to system reliability with order of magnitude accuracy would prove invaluable in terms of time and computational effort saved in the analysis of the reliability of a complex system.

It is the purpose of this thesis to investigate some simple approximations and the efficiency of these simple approximations with respect to established bounds and approximations. Sections II, III, and IV outline various notational conventions and definitions used in later sections. Section V

outlines some of the existing bounds and approximations while Section VI contains an investigation of some new approximations and an appraisal of their accuracy.

II. COHERENT SYSTEMS[1]

In general, the attention of this thesis will be focused on systems and components of the type commonly referred to as "Go/No Go." That is, at a particular point in time the system and its components are required to perform the function for which they were designed and, at that time, the components and system function properly (Go) or fail (No Go).

It is possible to write the following binary performance indicator for the system:

$$\phi = \begin{cases} 1 & \text{if the system functions properly,} \\ 0 & \text{if the system does not function properly (fails).} \end{cases}$$

The performance of the n components of the system can be indicated by the vector $\bar{x} = (x_1, x_2, \ldots, x_n)$ where

$$x_i = \begin{cases} 1 & \text{if component } i \text{ functions,} \\ 0 & \text{if component } i \text{ fails.} \end{cases}$$

Assuming that the system performance depends deterministically on component performance, it is possible to express the system performance indicator ϕ as a function of the component performance indicator $\bar{x}$. The function $\phi(\bar{x})$ is the <u>structure function</u> of order n.

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[1] Portions of Sections II, III, and IV have been paraphrased from Esary and Proschan (1962), Mine (1959), and Moore and Shannon (1956).

Three assumptions are made regarding system and component performance. First, it is reasonable to assume that if all of the components of a system function properly, the system will function properly. Second, it is also reasonable to assume that if none of the components work, the system will not work. Finally, it is assumed that if the system functions for a given set of functioning components, the system will not fail if the set of functioning components is enlarged. These assumptions can be expressed in the following manner:

i) $\phi(\bar{1}) = 1$ where $(\bar{1}) = (1,1,\ldots,1)$

ii) $\phi(\bar{0}) = 0$ where $(\bar{0}) = (0,0,\ldots,0)$

iii) $\phi(\bar{x}) \geq \phi(\bar{y})$ for $\bar{x} \geq \bar{y}$ (i.e. $x_i \geq y_i$, $i = 1,2,\ldots,n$).

A system whose structure function $\phi(\bar{x})$ has the above properties is defined to be a <u>coherent system</u>.

III. PATHS AND CUTS

Each specification $\bar{x} = (x_1, x_2, \ldots, x_n)$ of component performances determines a partition $\{A,B\}$ of $\{1,2,\ldots,n\}$ where

$$A = \{i \mid x_i = 1\} \quad \text{and} \quad B = \{i \mid x_i = 0\}.$$

If $\phi(\bar{x}) = 1$, A is called a path of the system and if $\phi(\bar{x}) = 0$, B is called a cut of the system. Occasionally, $\bar{x}$ is called a path if $\phi(\bar{x}) = 1$ and a cut if $\phi(\bar{x}) = 0$. The size of a path or cut is the number of components in the path or cut, but reference is sometimes made to the size of $\bar{x}$ given by

$$s(\bar{x}) = \sum_{i=1}^{n} x_i$$

Because of possible redundancy at the component and subsystem level, it is possible that not all of the components of A are, in fact, required to perform to insure that the system performs. Similarly, failure of all of the components of B may not be required to insure that the system fails. For these reasons, a path A is defined to be a minimal path if no proper subset of A is also a minimal path and a cut B is defined to be a minimal cut if no proper subset of B is also a minimal cut. Since the number of components in a system is assumed to be finite, each coherent system has a finite number of minimal paths $A_1, A_2, \ldots, A_a$ and a finite number of minimal cuts $B_1, B_2, \ldots, B_b$.

Consider a system in which at least one component has failed in each of the minimal paths with the exception of

minimal path A_j which has had no component failures. Then, all of the components in A_j must function if the system is to function and the components of A_j can be envisioned as being arranged in a logical series manner. Consider also a system in which at least one component is working in each of the minimal cuts with the exception of minimal cut B_k. Then, the system will fail if and only if every component in B_k fails and, for this reason, the components of B_k can be envisioned as being arranged in a logical parallel manner. As a result, structure functions for each minimal path A_j and each minimal cut B_k can be written as

$$\text{(III-1)} \qquad \alpha_j(\bar{x}) = \prod_{i \varepsilon A_j} x_i \ , \ j = 1,2,\ldots,a \ ,$$

and

$$\text{(III-2)} \qquad \beta_k(\bar{x}) = 1 - \prod_{i \varepsilon B_k} (1 - x_i) \ , \ k = 1,2,\ldots,b \ .$$

These structure functions are also binary performance indicators since

$$\alpha_j(\bar{x}) = \begin{cases} 1 & \text{if all the components of } A_j \text{ function,} \\ 0 & \text{otherwise,} \end{cases}$$

and,

$$\beta_k(\bar{x}) = \begin{cases} 0 & \text{if all the components of } B_k \text{ fail,} \\ 1 & \text{otherwise.} \end{cases}$$

Birnbaum, Esary, and Saunders (1961) have shown that it is possible to represent the system structure function as a logical parallel arrangement of all minimal path structure functions or as a logical series arrangement of all minimal cut structure functions. These minimal path and mininal cut representations are

$$\phi(\bar{x}) = 1 - \prod_{j=1}^{a} [1 - \alpha_j(\bar{x})] \tag{III-3}$$

and

$$\phi(\bar{x}) = \prod_{k=1}^{b} \beta_k(\bar{x}) \tag{III-4}$$

respectively.

IV. RELIABILITY

Let the binary random variable

$$X_i = \begin{cases} 1 & \text{if component } i \text{ functions,} \\ 0 & \text{if component } i \text{ fails.} \end{cases}$$

The <u>reliability of the ith component</u> is then

$$p_i = \Pr(X_i = 1) = E(X_i).$$

Let the binary variable

$$\phi(\bar{X}) = \begin{cases} 1 & \text{if the system functions,} \\ 0 & \text{if the system fails.} \end{cases}$$

The <u>reliability of the system</u> is then given by

$$h = \Pr[\phi(\bar{X}) = 1] = E[\phi(\bar{X})].$$

If the components perform independently, the distribution of $\bar{X} = (X_1, X_2, \ldots, X_n)$ is determined by $\bar{p} = (p_1, p_2, \ldots, p_n)$ and $h = h(\bar{p})$ is called the <u>reliability function</u> of the system. If $p_i = p$, $i = 1, 2, \ldots, n$, the reliability function is written as $h(p)$.

V. EXISTING BOUNDS AND APPROXIMATIONS

With the growth in size and complexity of systems, it has become increasingly difficult to compute the actual reliability of a system. Consequently, a number of bounds and approximations to the exact system reliability have been developed. Although bounds and approximations do not provide the true reliability of a system, the error introduced is more than justified in terms of computational time and effort saved.

Birnbaum, Esary, and Saunders (1961) have derived bounds on system reliability based on the length and width of a system. The length, ℓ, of the system is the minimum number of components whose functioning implies the system functions. The width, w, of the system is the minimum number of components whose failure implies the system fails. Letting

D_j = Number of paths of $\phi(\bar{x})$ such that $s(\bar{x}) = j$,

D_j^* = Number of cuts of $\phi(\bar{x})$ such that $s(\bar{x}) = j$,

and assuming a coherent system of order n with identical component reliabilities, the authors have shown that

$$\text{(V-1)} \qquad \sum_{i=\ell}^{n} D_i p^i (1 - p)^{n-i} = h(p) = 1 - \sum_{i=0}^{n-w} D_i^* p^i (1 - p)^{n-i}$$

Taking the first term only of the left hand side and the last term only of the right hand side results in the following inequality:

$$\text{(V-2)} \qquad D_\ell p^\ell (1-p)^{n-\ell} \le h(p) \le 1 - D^*_{n-w} p^{n-w} (1-p)^w .$$

Using

$$D_i \ge \frac{\binom{n}{i}}{\binom{n}{\ell}} D_\ell ,\ i = \ell, \ell+1, \ldots, n ,$$

and

$$D^*_i \ge \frac{\binom{n}{i}}{\binom{n}{w}} D^*_{n-w} ,\ i = 0, 1, \ldots, n-w$$

in (V-1) the upper and lower bounds on system reliability are

$$\text{(V-3)} \qquad 1 - \frac{D^*_{n-w}}{\binom{n}{w}} \sum_{i=0}^{n-w} \binom{n}{i} p^i (1-p)^{n-i}$$

and

$$\text{(V-4)} \qquad \frac{D_\ell}{\binom{n}{\ell}} \sum_{i=\ell}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

respectively.

Esary and Proschan (1963) have derived bounds on system reliability using the minimal path and minimal cut representations of the system structure. In both the minimal path representation, (III-3), and the minimal cut representation, (III-4), the same component may appear in more than one minimal path or minimal cut and it is necessary to imagine that all replications of the same component fail simultaneously. If, instead, each replica of the same component in the minimal path representation is replaced with an independent version of the same reliability, the expectation of (III-3) provides an upper bound to system reliability. Using a similar procedure on (III-4) results in a lower bound to system reliability. Esary and Proschan have shown that, for a given $\bar{p}$, the upper

bound, called the minimal path upper bound, is

$$(V\text{-}5)\qquad 1 - \prod_{j=1}^{a} (1 - \prod_{i \varepsilon A_j} p_i)$$

and that the lower bound, called the minimal cut lower bound, is

$$(V\text{-}6)\qquad \prod_{k=1}^{b} [1 - \prod_{i \varepsilon B_k} (1 - p_i)] .$$

Messinger and Shooman (1967) derived bounds to system reliability in terms of minimal tie sets and minimal cut sets. (Minimal tie sets and minimal cut sets are synonomous with the minimal paths and minimal cuts previously discussed.) Using the inclusion-exclusion method to write the probability statement for the union of several events, they have shown that the reliability of a system can be expressed, in terms of minimal cut sets, as

$$(V\text{-}7)\qquad R = 1 - \sum_{i=1}^{m} Pr(\bar{C}_i) + \sum_{\substack{i,j=1 \\ i<j}}^{m} Pr(\bar{C}_i \bar{C}_j) - \sum_{\substack{i,j,k=1 \\ i<j<k}}^{m} Pr(\bar{C}_i \bar{C}_j \bar{C}_k) \cdots (-1)^m Pr(\bar{C}_1 \bar{C}_2 \cdots \bar{C}_m)$$

where m represents the number of minimal cut sets of the system, $Pr(\bar{C}_i)$ denotes the probability that all the components in minimal cut set i fail, $Pr(\bar{C}_i \bar{C}_j)$ denotes the pairwise joint probability that all the components in minimal cut sets i and j fail, and so on. A lower bound to system reliability is given by

$$(V\text{-}8)\qquad R_{LB} = 1 - \sum_{i=1}^{m} Pr(\bar{C}_i)$$

and an upper bound by

$$\text{(V-9)} \qquad R_{UB} = 1 - \sum_{i=1}^{m} Pr(\bar{C}_i) + \sum_{\substack{i,j=1 \\ i<j}}^{m} Pr(\bar{C}_i \bar{C}_j)$$

Both (V-8) and (V-9) are accurate approximations for systems composed of highly reliable components.

It was noted by the authors that (V-8) and (V-9) are true in general and therefore can be applied to a system with dependent component failures. They have shown that by adding successive terms to (V-8) in the manner of (V-7), successive upper and lower bounds to system reliability can be computed. However, each successive upper bound is not necessarily an improvement over the previous upper bound and the same is true of the successive lower bounds.

Messinger and Shooman also presented an approximation called the Shannon approximation which is applicable to systems with highly reliable components. Let h_i equal the number of components in cut set i and let $h = \min_i \{h_i\}$. The Shannon approximation is then

$$\text{(V-10)} \qquad R_{SAH} = 1 - \sum_{i \ni h_i = h} Pr(\bar{C}_i)$$

If all components are independent and identical in reliability ($p_i = p$, $i = 1,2,\ldots,n$) and if each cut set contains exactly h components, then

$$R_{SAH} = 1 - m(1 - p)^h$$

Additional inequalities for reliability functions have been derived by Birnbaum and Esary (1965).

For highly complex systems the bounds and approximations discussed provide a means for evaluating the reliability of systems having reliability functions which are difficult to evaluate exactly. However, the computational effort required to determine the bounds or approximations may be excessive. In fact, it may not be possible to readily identify all of the minimal paths and minimal cuts of a complex system. Should this be the case, it would certainly be desirable to be able to make predictions about the true reliability of the system based on available minimal path and minimal cut information. Furthermore, the need for a relatively simple and reasonably accurate approximation to system reliability often exists in the preliminary analysis of system reliability or because of time constraints.

Esary, Marshall, and Proschan (1969) have suggested a method for approximating the reliability of a system composed of unreliable components with the reliability of the most reliable minimal paths of the system. System duality[2] suggests the possibility of approximating the reliability of a complex system composed of highly reliable components in terms of the least reliable minimal cuts of the system. This possibility is investigated in the following section by first analyzing the properties of minimal path and minimal cut approximations, extending the analysis to approximations based on the least

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[2] See Birnbaum, Esary, and Saunders (1961) for a discussion of duality.

reliable minimal cuts and most reliable minimal paths, and then analyzing the form of the approximating functions with respect to the system reliability functions.

VI. SOME SIMPLE APPROXIMATIONS

A. MINIMAL PATH AND MINIMAL CUT APPROXIMATIONS

Recall that for each minimal path A_j there exists a structure function $\alpha_j(\bar{x}) = \prod_{i \varepsilon A_j} x_i$ and for each minimal cut B_k there exists a structure function $\beta_k(\bar{x}) = 1 - \prod_{i \varepsilon B_k} (1 - x_i)$. Clearly, these minimal path and minimal cut structure functions are coherent. The reliability function for each minimal path is given by

$$\text{(VI-1)} \qquad h_j^A(\bar{p}) = E[\alpha_j(\bar{X})] = \prod_{i \varepsilon A_j} p_i$$

and the reliability function for each minimal cut is

$$\text{(VI-2)} \qquad h_k^B(\bar{p}) = E[\beta_k(\bar{X})] = 1 - \prod_{i \varepsilon B_k} (1 - p_i).$$

Thus, for a given $\bar{p} = (p_1, p_2, \ldots, p_n)$ the reliability of each minimal path, h_j^A , $j = 1,2,\ldots,a$, and the reliability of each minimal cut, h_k^B , $k = 1,2,\ldots,b$, can be computed.

If it is not possible to identify all of the minimal paths and minimal cuts of a complex coherent structure, it would certainly be desirable to use as much of the available information as possible in approximating system reliability. Since the system structure can be represented by the parallel arrangement of all minimal paths or the series arrangement of all minimal cuts, it is possible that the system reliability can be approximated by the reliability of the parallel

arrangement of those minimal paths which can be identified or by the series arrangement of those minimal cuts which can be identified.

Let $\{A_1, A_2, \ldots, A_a\}$ be the set of all minimal paths of a coherent system. If $P \subseteq \{A_1, A_2, \ldots A_a\}$ represents the set of minimal paths which can be identified, then the structure function of the parallel arrangement of the identified minimal paths is

$$\text{(VI-3)} \qquad \phi^P(\bar{x}) = 1 - \prod_J [1 - \alpha_j(\bar{x})]$$

where $J = \{j \mid A_j \varepsilon P\}$. It is also possible to write the structure function of the series arrangement of the identified minimal cuts as

$$\text{(VI-4)} \qquad \phi^C(\bar{x}) = \prod_K \beta_k(\bar{x})$$

where $K = \{k \mid B_k \varepsilon C\}$ and $C \subseteq \{B_1, B_2, \ldots, B_b\}$ is the set of identified minimal cuts. For a given $\bar{p}$, the reliability of the parallel arrangement of identified minimal paths can be computed using

$$\text{(VI-5)} \qquad h^P(\bar{p}) = E[\phi^P(\bar{X}) \mid \bar{p}]$$

and the reliability of the series arrangement of identified minimal cuts can be computed using

$$\text{(VI-6)} \qquad h^C(\bar{p}) = E[\phi^C(\bar{X}) \mid \bar{p}].$$

<u>Proposition 1</u>

If $P \subset P^* \subset \{A_1, A_2, \ldots, A_a\}$, then $h^P \leq h^{P^*} \leq h$, and if $C \subset C^* \subset \{B_1, B_2, \ldots, B_b\}$, then $h^C \geq h^{C^*} \geq h$.

Proof

The minimal path representation of the system structure function is $\phi(\bar{x}) = 1 - \prod_{j=1}^{a} [1 - \alpha_j(\bar{x})]$. If $P \subset P^* \subset \{A_1, A_2, \ldots, A_a\}$, then $1 - \prod_{J}[1 - \alpha_j(\bar{x})] \leq 1 - \prod_{J^*}[1 - \alpha_j(\bar{x})] \leq \phi(\bar{x})$ where $J = \{j | A_j \varepsilon P\}$ and $J^* = \{j | A_j \varepsilon P^*\}$. Taking expectations, $E\{1 - \prod_{J}[1 - \alpha_j(\bar{X})]\} \leq E\{1 - \prod_{J^*}[1 - \alpha_j(\bar{X})]\} \leq E[\phi(\bar{X})]$, i.e. for a given $\bar{p}$, $h^P \leq h^{P^*} \leq h$. The minimal cut representation of the system structure function is $\prod_{k=1}^{b} \beta_k(\bar{x})$. If $C \subset C^* \subset \{B_1, B_2, \ldots, B_b\}$, then $\prod_{K} \beta_k(\bar{x}) \geq \prod_{K^*} \beta_k(\bar{x}) \geq \phi(\bar{x})$ where $K = \{k | B_k \varepsilon C\}$ and $K^* = \{k | B_k \varepsilon C^*\}$. Taking expectations, $E[\prod_{K} \beta_k(\bar{X})] \geq E[\prod_{K^*} \beta_k(\bar{X})] \geq E[\phi(\bar{X})]$, i.e. for a given $\bar{p}$, $h^C \geq h^{C^*} \geq h$.

Note that if the set P consists of a single identified minimal path and if the set C consists of a single identified minimal cut, upper and lower bounds on system reliability are easily established by computing the reliabilities of the minimal path and minimal cut using (VI-1) and (VI-2) respectively. If either P or C is the entire set of minimal paths or minimal cuts, the exact reliability of the system can be computed using (VI-5) or (VI-6) as appropriate.

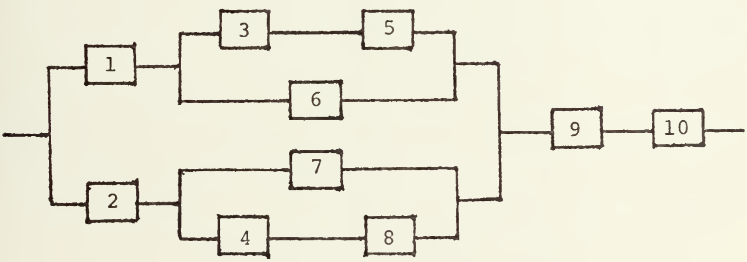
In general, Proposition 1 states that the conservative estimate of system reliability using minimal paths will not be decreased and the optimistic estimate of system reliability

will not be increased as additional minimal paths or cuts are included in the approximations. In fact, it can be shown that if $0<p_i<1$, $i = 1,2,\ldots,n$, all of the inequalities of Proposition 1 are strict and an improvement in estimates is guaranteed as additional minimal paths or minimal cuts are included in the approximations.

The following example illustrates the applicability of Proposition 1.

Example 1: Communications System

The system under consideration is an operating tropospheric-scatter radio communications system. The reliability block diagram and component descriptions of the receive portion of the system are shown below.

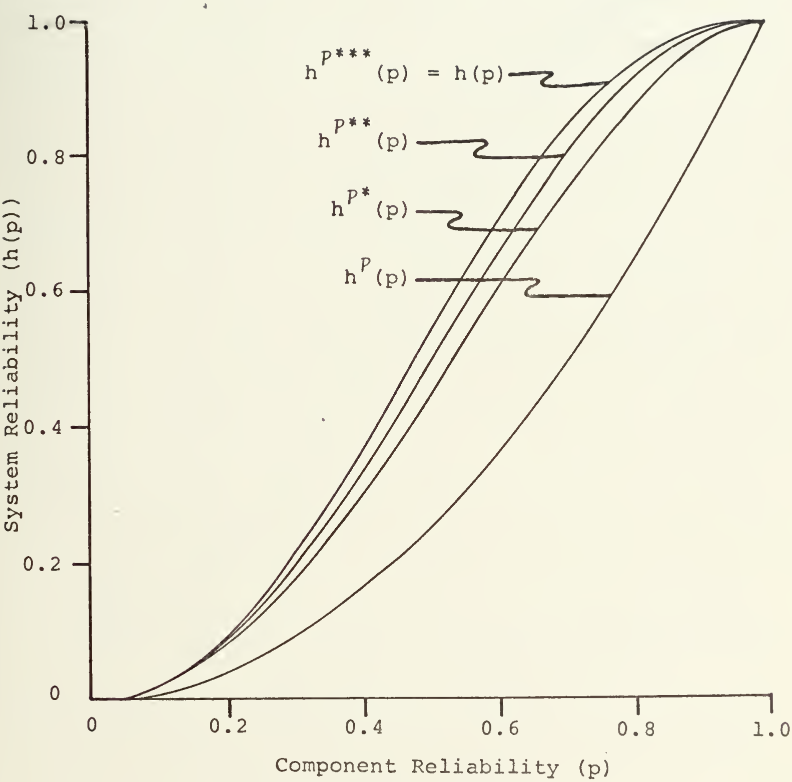


Component Number	Description
1,2	Antenna
3,4	Duplexer (Separates transmitted and received signals)
5,6,7,8	Radio Receiver
9	Demodulation Equipment
10	Circuit Conditioning Equipment (Amplifiers, attenuators, etc.)

For the purposes of this example, it is sufficient to consider the input and output of the system as a single voice circuit. Because of extreme redundancy, the reliability of components nine and ten is nearly unity and these components are removed from the system for this example. Realistically, each of the components is highly reliable and it is assumed that $p_i = p$, $i = 1,2,\ldots,8$. Under this assumption, the reliability function for the mosified system is

$$h(p) = 2p^2 + 2p^3 - 3p^4 - 2p^5 + p^6 + 2p^7 - p^8 .$$

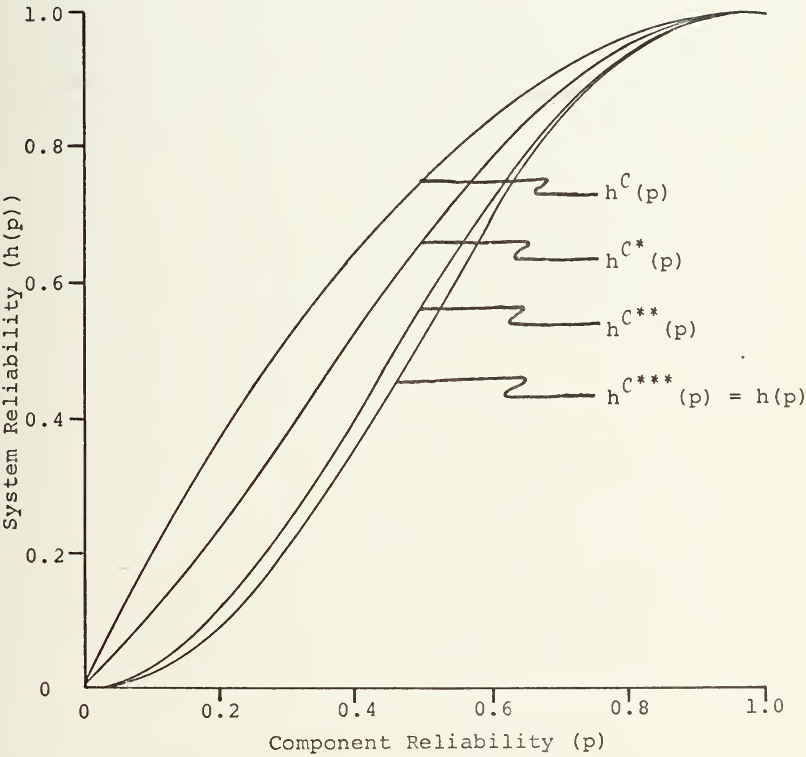
The minimal paths are $A_1 = \{1,6\}$, $A_2 = \{2,7\}$, $A_3 = \{1,3,5\}$ and $A_4 = \{2,4,8\}$. The minimal cuts are $B_1 = \{1,2\}$, $B_2 = \{1,4,7\}$, $B_3 = \{1,7,8\}$, $B_4 = \{2,3,6\}$, $B_5 = \{2,5,6\}$, $B_6 = \{3,4,6,7\}$, $B_7 = \{3,6,7,8\}$, $B_8 = \{4,5,6,7\}$, and $B_9 = \{5,6,7,8\}$. Figures 1 and 2 illustrate the improvement in conservative and optimistic estimates of system reliability as additional minimal paths and minimal cuts are included in the approximation. When investigating the reliability of a symmetric system (such as the communications system of this example) in terms of minimal cuts, certain engineering considerations should also be included in the reliability analysis. Note that minimal cuts B_2 and B_4 are symmetric in that they are identical in size and with respect to component types. Similarly, minimal cuts B_3 and B_5 are symmetric and minimal cuts B_7 and B_8 are symmetric. Then, any approximation to system reliability should include at least those minimal cuts which are symmetric with one another. For example, if minimal cut



$P = \{A_1\}$, $P^* = \{A_1, A_2\}$, $P^{**} = \{A_1, A_2, A_3\}$,

$P^{***} = \{A_1, A_2, A_3, A_4\}$

Figure 1: Minimal Path Approximations to Communications System Reliability



$C = \{B_1\}$, $C^* = \{B_1, B_2, B_3\}$, $C^{**} = \{B_1, B_2, \ldots, B_5\}$,

$C^{***} = \{B_1, B_2, \ldots, B_9\}$

Figure 2: Minimal Cut Approximations to Communications System Reliability

B_2 is used in an approximation, minimal cut B_4 should also be used in that approximation; if minimal cuts B_2, B_3, and B_4 are used in an approximation, minimal cut B_5 should also be used in that approximation.

It is interesting to note that if h^C is used as an optimistic estimate of system reliability for a given $\bar{p}$, it is possible to compute a lower bound, denoted h_I^C, for h^C. The method used to compute the lower bound is the method used by Esary and Proschan to compute the minimal cut lower bound for system reliability. That is, if any components are common to two or more of the minimal cuts used in the approximation, $h^C(\bar{p})$, these common components are replaced with independent versions of identical reliability. The lower bound is then computed using

$$\text{(VI-7)} \qquad h_I^C = \pi_K [1 - \pi_{i \varepsilon B_k} (1 - p_i)]$$

where $K = \{k | B_k \varepsilon C\}$. Since it is not necessary to expand the structure function $\phi^C(\bar{x})$ in computing h_I^C, h_I^C is much easier to compute than h^C. If h^P is used as a conservative estimate of system reliability, it is similarly possible to compute an upper bound, denoted h_I^P, to h^P. The upper bound is given by

$$\text{(VI-8)} \qquad h_I^P = 1 - \pi_J (1 - \pi_{i \varepsilon A_j} p_i)$$

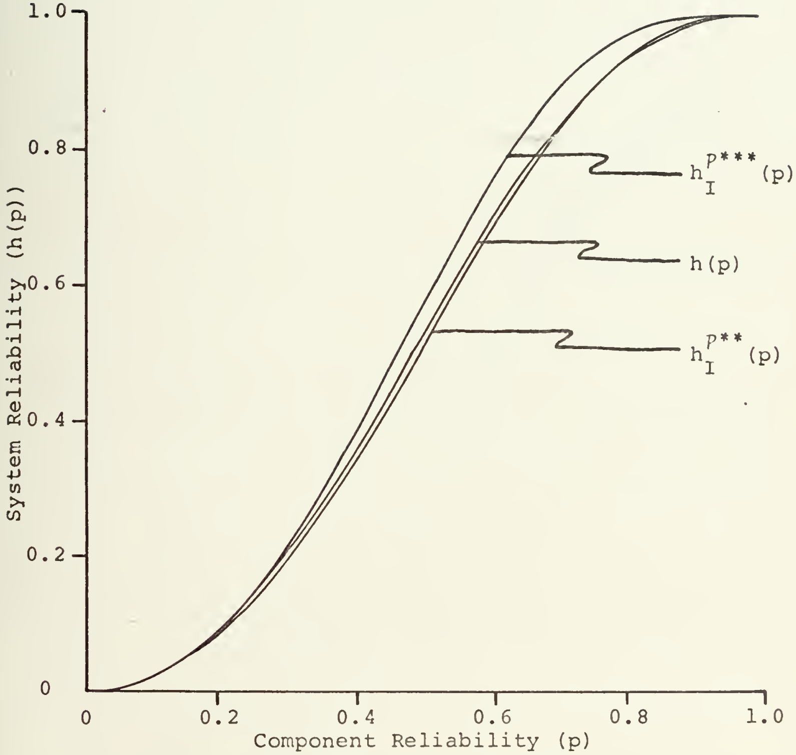
where $J = \{j | A_j \varepsilon P\}$. The computational effort required for h_I^P is also less than that required for h^P.

If P is the set of all minimal paths of the system, the upper bound, h_I^P, is the minimal path upper bound for system reliability. If C is the set of all minimal cuts of the

system, the lower bound, h_I^{C} , is the minimal cut lower bound for system reliability.

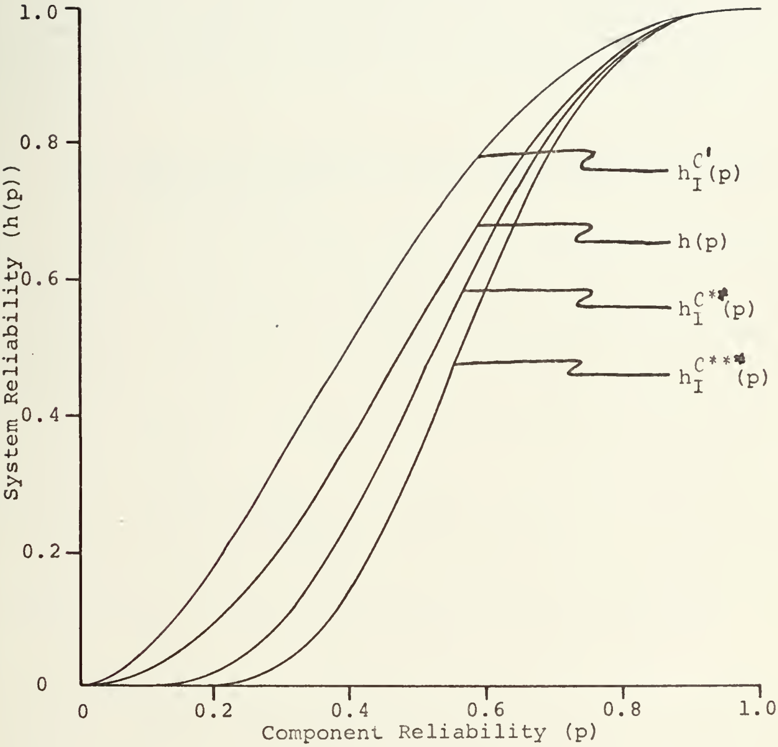
Since h^{C} is an optimistic estimate of system reliability and h_I^{C} is a lower bound to h^{C} , it is possible that h_I^{C} might be useful as an estimate of system reliability. Since h^{P} is a conservative estimate of system reliability and h_I^{P} is an upper bound to h^{P} , it is also possible that h_I^{P} could be used as an estimate of system reliability.

This technique of approximating system reliability by assuming the independence of minimal paths and the independence of minimal cuts was applied to the system described in Example 1. Estimates, h_I^{P} and h_I^{C} , to system reliability were computed for several subsets of the set of all minimal paths and for several subsets of the set of all minimal cuts. For all subsets P and C considered and for $p \geq 0.90$, the estimates h_I^{C} were accurate to within $\pm$ 0.005 while the estimates h_I^{P} were accurate to within $\pm$ 0.025. The results for $P^{**} = \{A_1, A_2, A_3\}$ and $P^{***} = \{A_1, A_2, A_3, A_4\}$ are shown in Figure 3 while the results for $C' = \{B_1, B_2\}$, $C^{**} = \{B_1, B_2, \ldots, B_5\}$, and $C^{***} = \{B_1, B_2, \ldots, B_9\}$ are shown in Figure 4. Note that $h_I^{P^{***}}(p)$ is the minimal path upper bound for system reliability and that $h_I^{C^{***}}(p)$ is the minimal cut lower bound for system reliability.



$P^{**} = \{A_1, A_2, A_3\}$, $P^{***} = \{A_1, A_2, A_3, A_4\}$

Figure 3: Independent Minimal Path Approximation and Bound to Communications System Reliability



$C' = \{B_1, B_2\}$, $C** = \{B_1, B_2, \ldots, B_5\}$, $C*** = \{B_1, B_2, \ldots, B_9\}$

Figure 4: Independent Minimal Cut Approximations and Bound to Communications System Reliability

These simple approximations might prove useful in the preliminary analysis of the system reliability. It should be pointed out, however, that it is impossible to establish a general ordering of the magnitudes of h_I^C , h , and h_I^P and that the accuracy of the estimates is sensitive to the size of the subsets P and C (unlike the results of Proposition 1, the inclusion of additional minimal paths or minimal cuts in computing the estimates does not guarantee an improvement in accuracy).

B. MOST RELIABLE MINIMAL PATH AND LEAST RELIABLE MINIMAL CUT APPROXIMATIONS

Assume, for the moment, that it is possible to identify all of the minimal cuts and minimal paths of a system. For a given $\bar{p}$, it is possible to order the minimal path reliabilities such that

$$h^A_{(1)} = \min_j h^A_j \ , \ j = 1,2,\ldots,a \ ,$$

$$h^A_{(j)} \le h^A_{(j+1)} \ , \ j = 1,2,\ldots,a-1 \ ,$$

$$h^A_{(a)} = \max_j h^A_j \ , \ j = 1,2,\ldots,a \ ,$$

and to order the minimal cut reliabilities such that

$$h^B_{(1)} = \min_k h^B_k \ , \ k = 1,2,\ldots,b \ ,$$

$$h^B_{(k)} \le h^B_{(k+1)} \ , \ k = 1,2,\ldots,b-1 \ ,$$

$$h^B_{(b)} = \max_k h^B_k \ , \ k = 1,2,\ldots,b \ .$$

Define a <u>Most Likely Minimal Path</u> (MLMP) as a minimal path A_p such that $h^A_p = h^A_{(a)}$ and define a <u>Most Vulnerable Minimal Cut</u> (MVMC) as a minimal cut B_c such that $h^B_c = h^B_{(1)}$. The MLMP and

MVMC may not be unique. Therefore, let r denote the multiplicity of MLMP's, i.e. the number of minimal paths where $h^A_{(a)} = h^A_{(a-1)} = \cdots = h^A_{(a-r+1)}$, and let s denote the multiplicity of MVMC's, i.e. the number of minimal cuts where $h^B_{(1)} = h^B_{(2)} = \cdots = h^B_s$.

Proposition 2

For a coherent system of order n and a given $\bar{p}$, $0<p_i<1$, $i = 1,2,\ldots,n$, the best, i.e. minimum error, conservative (optimistic) estimate of system reliability using a single minimal path (cut) is the reliability of the MLMP (MVMC).

Proof

For $k = 1,2,\ldots,b$, $\beta_k(\bar{x}) \geq \phi(\bar{x})$. Given $\bar{p}$, $E[\beta_k(\bar{X})] \geq E[\phi(\bar{X})]$, or, $h^B_k \geq h$. By definition, $h^B_c = \min_k h^B_k$ and, therefore, Error $= |h^B_k - h|$ is a minimum for k = c. The proof for the MLMP is similar.

From the foregoing, it can be seen that if it is not possible to identify all of the minimal cuts or paths of a coherent system, the best optimistic estimate of system reliability using a single minimal cut is the reliability of the least reliable identified minimal cut and the best conservative estimate of system reliability using a single minimal path is the reliability of the most reliable identified minimal path.

If it is possible to identify all of the minimal paths and minimal cuts, if r>1 and/or s>1, and if $0<p_i<1$, $i = 1,2,\ldots,n$, Proposition 1 then guarantees that an approximation which uses

all of the MLMP's or MVMC's is a better approximation than one which uses only a single MLMP or MVMC. If the above conditions are satisfied, the conservative estimate can be extended to

$$\text{(VI-9)} \qquad h^r = E[\phi^{P}(\bar{x})\,|\,\bar{p}]$$

where $P = \{A_j \mid h_j^A = h_p^A\}$ and $\phi^{P}(\bar{x})$ is defined by (VI-3) and the optimistic estimate can be extended to

$$\text{(VI-10)} \qquad h^s = E[\phi^{C}(\bar{x})\,|\,\bar{p}]$$

where $C = \{B_k \mid h_k^B = h_c^B\}$ and $\phi^{C}(\bar{x})$ is defined by (VI-4).

If the MLMP and the MVMC can be identified for a given system and given component reliabilities, a change in any of the component reliabilities may result in another minimal path (cut) being designated as the MLMP (MVMC). If $p_i = p$, $i = 1,2,\ldots,n$, the MLMP (MVMC) will be the same minimal path (cut) for any $0<p<1$. In fact, for the case of identical component reliabilities, the MLMP (MVMC) is that minimal path (cut) with a minimum number of components. In other words, the MLMP is the length, ℓ, of the system and the MVMC is the width, w, of the system.

Example 2: Communications System, Continued

Recall that the minimal paths for the modified system were $A_1 = \{1,6\}$, $A_2 = \{2,7\}$, $A_3 = \{1,3,5\}$, $A_4 = \{2,4,8\}$ and the minimal cuts were $B_1 = \{1,2\}$, $B_2 = \{1,4,7\}$, $B_3 = \{1,7,8\}$, $B_4 = \{2,3,6\}$, $B_5 = \{2,5,6\}$, $B_6 = \{3,4,6,7\}$, $B_7 = \{3,6,7,8\}$, $B_8 = \{4,5,6,7\}$, $B_9 = \{5,6,7,8\}$. If $p_i = p$, $i = 1,2,\ldots,n$, the MVMC is B_1 and the reliability function of the MVMC is

$h_c^B(p) = 1 - (1 - p)^2$. The MLMP is A_1 or A_2 and the reliability function of the MLMP is $h_p^A(p) = p^2$. Since the multiplicity of MLMP's is $r = 2$, a better conservative approximation to system reliability would be $h^r(p) = 1 - (1 - p^2)^2$ (note that paths A_1 and A_2 are mutually exclusive which may not always be true). The MVMC, MLMP, and MLMP multiplicity approximations to the actual communications system reliability are shown in Figure 5. For $p \geq 0.90$, h_c^B is accurate to within 0.004 and h^r is accurate to within 0.025.

An appreciation of the accuracy of the approximations outlined above for systems with highly reliable components can be gained by analyzing the form of the true reliability function and the approximating functions in the region of highly reliable components. By letting $p_i = p$, $i = 1,2,\ldots,n$ and using the results of Moore and Shannon (1956), Esary and Proschan (1962) have shown that

$$\frac{d}{dp} h(p)\Big|_{p=1} = \text{Number of cuts of size 1 in the system,}$$

and that

$$\frac{d}{dp} h(p)\Big|_{p=0} = \text{Number of paths of size 1 in the system.}$$

For a system with no paths or cuts of size 1, the reliability function of the system resembles an S and is said to be S-shaped.

The MLMP approximation to a system with no cuts of size 1 is $h_p^A(p) = p^\ell$ where ℓ is the length of the system structure. Now,

$$\frac{d}{dp} h_p^A(p)\Big|_{p=1} = \ell > 0$$

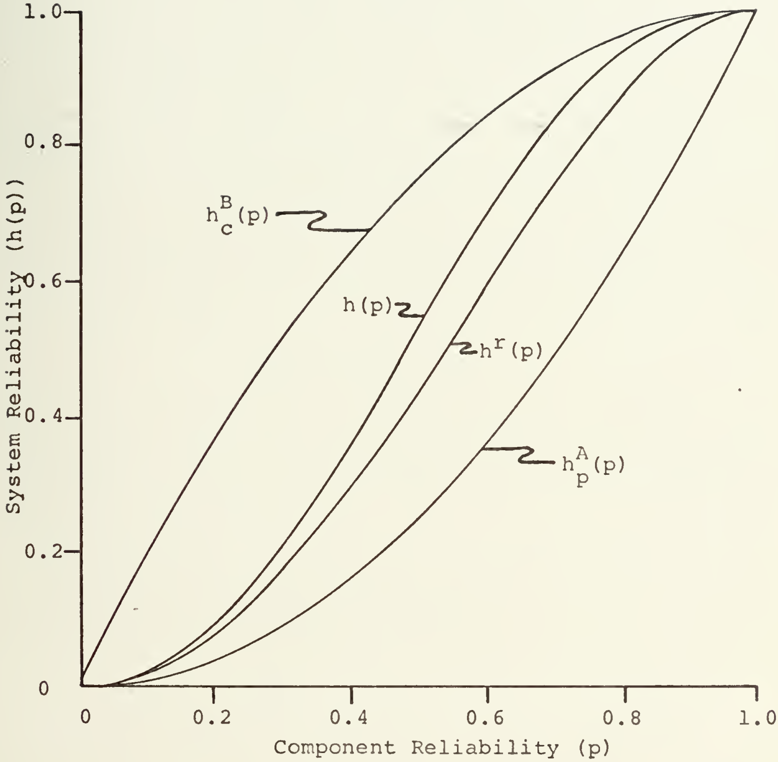


Figure 5: Most Vulnerable Minimal Cut, Most Likely Minimal Path and Most Likely Minimal Path Multiplicity Approximations to Communications System Reliability

and the MLMP approximation cannot be considered a good approximation in the region of highly reliable components. However, if $r>1$, or if the second MLMP (A_{p*} where $h^A_{p*} = h^A_{(a-1)}$) is included in the approximation, the condition that the system has no cuts of size 1 is sufficient to insure that

$$\frac{d}{dp} h^r(p)\bigg|_{p=1} = 0 \quad \text{and} \quad \frac{d}{dp} h^*(p)\bigg|_{p=1} = 0$$

where $h^r(p)$ is the MLMP multiplicity approximation and $h^*(p)$ is an approximation developed from the MLMP and the second MLMP. The MVMC approximation to the same system is $h^B_c(p) = 1 - (1-p)^w$ where w is the width of the system structure.

$$\frac{d}{dp} h^B_c(p)\bigg|_{p=1} = 0$$

and the MVMC approximation is considered to be an accurate approximation to system reliability for a system composed of highly reliable components.

The MLMP approximation to the reliability of a system with q cuts of size 1 is $h^A_p(p) = p^\ell$ and

$$\frac{d}{dp} h^A_p(p)\bigg|_{p=1} = \ell \geq q \ .$$

Although the MLMP approximation is more accurate for a system with q cuts of size 1 than for a system with no cuts of size 1, the accuracy of the approximation is clearly dependent on the magnitudes of q and ℓ. Including multiplicity of MLMP's or a second minimal path in the approximation would, in this case, decrease the accuracy of the approximation. Consideration might be given to the possible use of a linear function (such

as $h^*(p) = q(p - 1) + 1)$ to approximate the reliability of a system composed of highly reliable components and having q cuts of size 1. The MVMC approximation to a system with q cuts of size 1 is $h_c^B(p) = p$ and

$$\frac{d}{dp} h_c^B(p)\Big|_{p=1} = 0 \ .$$

Note, however, that the multiplicity of MVMC's is s = q. If no two of the q components are identical, the system reliability function can be approximated with $h^s(p) = p^q$ and

$$\frac{d}{dp} h^s(p)\Big|_{p=1} = q \ .$$

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13. ABSTRACT

Several reliability approximations for complex coherent systems with highly reliable components are reviewed and the need for some relatively simple approximations which would be useful in a preliminary reliability analysis of the same type of system is presented. Some approximations based on minimal paths, minimal cuts, most reliable minimal paths, and least reliable minimal cuts are investigated in terms of simplicity and accuracy. The possible applications of the approximations are illustrated through the use of examples.

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